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Worldsheet correlators in $\mathrm{AdS}_3/\mathrm{CFT}_2$

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Abstract: The $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence is checked beyond the supergravity approximation by comparing correlation functions. To this end we calculate 2- and 3-point functions on the sphere of certain chiral primary operators for strings on $AdS_3 \times S^3 \times T^4$. These results are then compared with the corresponding amplitudes in the dual 2-dimensional conformal field theory. In the limit of small string coupling, where the sphere diagrams dominate the string perturbation series, beautiful agreement is found.

Keywords: AdS-CFT Correspondence, Gauge-gravity correspondence.

Contents

1. Introduction

Over the last few years, increasing evidence has been accumulated that (in the large N limit) 4-dimensional gauge theories have a dual description in terms of a higher-dimensional string theory. This so-called AdS/CFT correspondence [1 − 3] has predominantly been tested in the supergravity approximation of string theory at small curvatures (see however the analysis in the plane-wave limit [4]). On the other hand, the AdS/CFT correspondence actually relates the weak coupling regime of gauge theory to backgrounds with high curvature in string theory. It would therefore be very interesting to understand and check the correspondence for such configurations. Unfortunately, the high curvature regime of type IIB string theory on $AdS_5 \times S^5$ that is dual to four-dimensional $\mathcal{N} = 4$ SU(N) SYM theory is difficult to control at present.

The situation is different for the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality for which both the ten-dimensional string theory as well as the two-dimensional boundary conformal field theory are explicitly known (for a review see [5]). Starting from the intersection of N_1 D1-branes and N_5 D5-branes compactified on a four-torus T^4, one finds $AdS_3 \times S^3 \times T^4$ as the corresponding near-horizon geometry. After S-duality, the worldsheet theory with target space $AdS_3 \times S^3 \times T^4$ is an $\mathcal{N}=1$ SL(2)$\times$SU(2) WZW model which is in principle solvable to all orders in α'. The conjectured dual field theory arises as the infrared fixed point theory living on the D1-D5 system. The fixed point can be described by an $\mathcal{N}=(4,4)$ sigma model whose target space is a deformation of the symmetric product orbifold $\mathrm{Sym}(\tilde{T}^4)^N = (\tilde{T}^4)^N/S_N$, where $\tilde{T}^4$ is related to the four-torus T^4 [6] and S_N is the permutation group of N symbols. It has been argued in [7] that the orbifold limit corresponds to the point $(N_1, N_5)=(N,1)$ in the D1-D5 moduli space.

In this paper, we will subject this correspondence to a non-trivial test by comparing the correlators of certain chiral primary fields in both theories. We will follow the approach outlined in [8] that explains how the correlators of the boundary conformal field theory can be obtained from the world-sheet correlators of the string theory.[1] There it is suggested that the integration of vertex operators over the worldsheet yields the corresponding boundary operator. This is motivated by the fact that the continuous SL(2) representation labels $x, \bar{x}$ are identified with the complex coordinates in the boundary conformal field theory, as already noted in [10].

Here we shall apply this programme to the full theory, including the SU(2) WZW model as well as the theory on T^4. We shall concentrate on the chiral primary operators that correspond to the n-cycle twist operators in the boundary conformal field theory [11]. In the boundary conformal field theory their correlation functions were first calculated for a special case in [12] and then, in general, in [13, 14] (see also [15] for earlier work). The correlation functions in the world-sheet theory (on the sphere) can be reduced to standard 3-point functions of the SU(2) and SL(2) WZW models that have been determined before in [16, 17] and [18 – 20], respectively. The comparison of the 2-point functions determines the relative normalisation constants between the field operators on both sides of the correspondence. The comparison of the 3-point functions is then a non-trivial consistency check. In the large N limit (in which the dominant string contribution comes from the sphere diagram) we find beautiful agreement.

In the supergravity approximation, the correlators of these chiral primary fields were previously computed in [21, 22]. A quantitative comparison with the boundary conformal field theory at the orbifold point (for which the conformal field theory correlators are known) is however impeded by the fact that the corresponding D1-D5 system is at strong string coupling. By S-duality this can be mapped to the F1-NS5 system at small string coupling, but then the AdS_3 radius is of order of the string length, and hence α' corrections have to be taken into account.

The paper is organised as follows. In section 2, we review the symmetric orbifold theory and the correlators of n-cycle twist operators found by Lunin and Mathur [13, 14]. In section 3, we then compute correlation functions of the corresponding primary vertex operator in the worldsheet theory with target space $AdS_3 \times S^3 \times T^4$ and compare them with those in the boundary conformal field theory. Our conclusions are contained in section 4.

[1] For correlation functions involving more than three fields, the analysis of Gopakumar [9] should be relevant.

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There are three appendices where some of the more technical material is explained.

2. Sigma model on the symmetric orbifold

The Higgs branch of the D1-D5 system compactified on T^4 flows in the infrared to a two-dimensional $(4,4)$ superconformal field theory with conformal charge $c = 6N_1N_5$. The global part of the $\mathcal{N} = (4,4)$ superconformal algebra forms the supergroup $\mathrm{SU}(1,1|2)_L \times \mathrm{SU}(1,1|2)_R$ and contains the R-symmetry group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. It is generally believed that the target space of this particular $(4,4)$ SCFT is identical to the instanton moduli space $\mathcal{P}$ of N_1 instantons in the $\mathrm{U}(N_5)$ gauge theory on T^4. This idea is motivated by the fact that the D1-branes can be viewed as instantons of the low-energy theory of the D5-branes. The appropriate low-energy description of the D1-D5 system is therefore given by a sigma model with target space $\mathcal{P}$.

2.1 The n-cycle twist operators

The moduli space $\mathcal{P}$ of the infrared $(4,4)$ SCFT is the symmetric product orbifold $\mathrm{Sym}(\tilde{T}^4)^N = (\tilde{T}^4)^N/S_N$, where S_N denotes the permutation group of $N = N_1N_5$ symbols. The four-torus $\tilde{T}^4$ is closely related to the T^4 in the worldsheet model, for the exact relation see ref. [6]. The moduli space is parametrized by the scalars X^i_A, where the $A = 1,\ldots,N$ runs over the N copies of $\tilde{T}^4$, and $i = 1,2,3,4$ denotes the vector index in $\tilde{T}^4$. The orbifold action on $(\tilde{T}^4)^N$ by the symmetric group S_N means that any point $(X_1,\ldots,X_N)$ on $(\tilde{T}^4)^N$ is identified with the points obtained by any permutation of the X_A $(A = 1,\ldots,N)$.

The orbifold theory consists of the invariant operators of the original theory, together with operators from the twisted sectors. For non-abelian orbifolds the twisted sectors are associated to the conjugacy classes of the orbifold group, which is S_N in our case. These conjugacy classes are labelled by partitions of N into positive integers,

$$\sum_{l=1}^{N} l k_l = N = N_1 N_5 \,, \tag{2.1}$$

corresponding to the permutations with k_l cycles of length l. We are mainly interested in the conjugacy class with one cycle of length n, i.e. $k_n = 1$ and $k_1 = N - n$. The corresponding permutations are of the form

$$(X_{A_1} \to X_{A_2}, \ldots, X_{A_n} \to X_{A_1})\,, \qquad X_B \to X_B\,, \quad B \notin \{A_1,\ldots,A_n\}\,, \tag{2.2}$$

where $A_1 \neq A_2 \neq \ldots \neq A_n \in \{1,\ldots,N\}$. We call the corresponding operators n-cycle twist operators and denote them by

$$\Sigma^{(n)}(x,\bar{x})\,, \qquad j' = \tfrac{n-1}{2} = 0, \tfrac{1}{2}, 1, \ldots, \tfrac{N-1}{2}\ . \tag{2.3}$$

The label $j' = \bar{j}' = (n-1)/2$ denotes the charge under the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ R-symmetry group, and their conformal dimension is $h = \bar{h} = (n-1)/2$. In particular, they therefore satisfy $h = j'$, and thus define chiral primary fields. These fields are the analogues of the

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single-trace operators in the $\mathcal{N}=4$ super Yang-Mills theory. As such, they correspond to single-particle states. The theory obviously also has operators for conjugacy classes with more than one cycle; these describe multi-particle states (see also [13, 14, 5]). In the following we shall however restrict attention to the above n-cycle twist operators.

2.2 Correlators of n-cycle twist operators

The chiral primaries of the $(4,4)$ superconformal field theory actually fall into short multiplets of the supergroup $\mathrm{SU}(1,1|2)_L\times\mathrm{SU}(1,1|2)_R$. In particular, they therefore define representations of the $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetry. The above operators $\Sigma^{(n)}$ are the highest weight states with respect to this action; if we include the m' and $\bar{m}'$ dependence we should therefore write them as $\Sigma^{(n)}\equiv\Sigma^{(n)}_{j',j'}$. More generally, we may thus also define the operators $\Sigma^{(n)}_{m',\bar{m}'}$ by acting on $\Sigma^{(n)}_{j',j'}$ with the $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ lowering operators — see [13] for more details. There are also different chiral primary operators that can be obtained by multiplying $\Sigma^{(n)}$ with appropriate combinations of spinors ψ^i_A [5], the superpartners of the X^i_A. However, in this paper we shall only consider the chiral primaries $\Sigma^{(n)}_{m',\bar{m}'}$.

The two- and three-point functions of the chiral twist operator $\Sigma^{(n)}_{m',\bar{m}'}$ have been calculated, using path integral methods, in [13, 14]. The two-point function is given by [13][2]

$$\langle\Sigma^{(n)}_{m',\bar{m}'}(x_1,\bar{x}_1)\Sigma^{(n)}_{-m',-\bar{m}'}(x_2,\bar{x}_2)\rangle=\frac{1}{|x_{12}|^{4h}}\,, \tag{2.4}$$

where $h=(n-1)/2$. Here we have normalised the fields in the usual way. With this normalisation the three-point function is given by

$$\langle\Sigma^{(n_1)}_{m_1',\bar{m}_1'}(x_1,\bar{x}_1)\,\Sigma^{(n_2)}_{m_2',\bar{m}_2'}(x_2,\bar{x}_2)\,\Sigma^{(n_3)}_{m_3',\bar{m}_3'}(x_3,\bar{x}_3)\rangle=\delta^2(\textstyle\sum_{a=1}^3 m_a')C_{n_1,n_2,n_3}\displaystyle\prod_{i<j}\frac{1}{|x_{ij}|^{2h_{ij}}}\,, \tag{2.5}$$

with $h_{12}=h_1+h_2-h_3$, etc., and δ^2 is the product of the Kronecker δ for the barred and unbarred $\sum_a m_a'$. For the choice $(d\geq 0)$

$$m_1'=\bar{m}_1'=j_1'-d\,,\qquad m_2'=\bar{m}_2'=j_2'\,,\qquad m_3'=\bar{m}_3'=-(j_1'+j_2'-d)=-j_3'\,, \tag{2.6}$$

the fusion coefficients $C_{n_1,n_2,n_3}=C^{(1)}_{n_1,n_2,n_3}C^{(2)}_{n_1,n_2,n_3}$ are given by

$$C^{(1)}_{n_1,n_2,n_3}=\frac{s^2\,d!\,(n_1-d-1)!}{n_1\,n_2\,n_3\,(n_1-1)!}\,,\quad C^{(2)}_{n_1,n_2,n_3}=\frac{\sqrt{n_1n_2n_3(N-n_1)!(N-n_2)!(N-n_3)!}}{(N-s)!\sqrt{N!}} \tag{2.7}$$

with $2s=n_1+n_2+n_3-1$ and $2d=n_1+n_2-n_3-1$. Here $C^{(1)}_{n_1,n_2,n_3}$ is the coefficient in the three-point function coming from single representatives of the conjugacy classes; the factor $C^{(2)}_{n_1,n_2,n_3}$ on the other hand results from the summation over all elements in the given conjugacy classes. Since n_j denotes the cycle length we obviously have $n_j\leq N$ for

[2] The twist operator $\Sigma^{(n)}$ is denoted by O_n^{--} in [13] and defined by eq. (6.42) therein.

$j = 1, 2, 3$. In addition, as explained in [13] we also have that $s \leq N$. For the special case $d = 0$, the three-point function (2.5) was first found in [12].

For later convenience, we also write the correlators in terms of the labels $j' = (n-1)/2$, identifying $V_{j',m',\bar{m}'} \equiv \Sigma^{(n)}_{m',\bar{m}'}$. As in eq. (2.4), we normalise the two-point function in the boundary conformal field theory as

$$\langle V_{j',m',\bar{m}'}(x_1,\bar{x}_1)\, V_{j',-m',-\bar{m}'}(x_2,\bar{x}_2)\rangle = \frac{1}{|x_{12}|^{4j'}} \ . \tag{2.8}$$

The corresponding three-point function then equals

$$\langle V_{j_1',m_1',\bar{m}_1'}(x_1,\bar{x}_1)\, V_{j_2',m_2',\bar{m}_2'}(x_2,\bar{x}_2)\, V_{j_3',m_3',\bar{m}_3'}(x_3,\bar{x}_3)\rangle = C^{\text{bcft}}_{j_1',j_2',j_3'} \prod_{i<j} \frac{1}{|x_{ij}|^{2j_{ij}'}} \ , \tag{2.9}$$

where $C^{\text{bcft}}_{j_1',j_2',j_3'} = C^{(1)}_{j_1',j_2',j_3'} C^{(2)}_{j_1',j_2',j_3'}$ is given by ($d = j_{12}' \geq 0$)

$$C^{(1)}_{j_1',j_2',j_3'} = (j_1'+j_2'+j_3'+1)^2 \left(\frac{\Gamma(j_{13}'+1)\Gamma(j_{12}'+1)}{\Gamma(2j_1'+1)\prod_{i=1}^{3}(2j_i'+1)}\right) \tag{2.10}$$

$$C^{(2)}_{j_1',j_2',j_3'} = \left(\frac{\prod_{i=1}^{3}(2j_i'+1)\Gamma(N-2j_i')}{\Gamma(N-j_1'-j_2'-j_3')^2\Gamma(N+1)}\right)^{1/2} \ . \tag{2.11}$$

Here the m' labels are chosen as in eq. (2.6) and $j_{12}' = j_1' + j_2' - j_3'$, etc.

As before, there are some restrictions on the quantum numbers j_i' ($i = 1, 2, 3$). In particular, the bounds $n_i = 2j_i' + 1 \leq N$ and $s \leq N$ in eq. (2.5) translate into

$$0 \leq j_i' \leq \tfrac{N-1}{2} \ , \qquad j_1' + j_2' + j_3' \leq N - 1 \ . \tag{2.12}$$

For the comparison with the string correlators on the sphere only the large N limit will be relevant; in this limit, the total coefficient simplifies to

$$C^{\text{bcft}}_{j_1',j_2',j_3'} \overset{N\to\infty}{=} \frac{(j_1'+j_2'+j_3'+1)^2}{\sqrt{N}\prod_i (2j_i'+1)^{\frac{1}{2}}} \frac{\Gamma(j_{13}'+1)\Gamma(j_{12}'+1)}{\Gamma(2j_1'+1)} \ . \tag{2.13}$$

In the next section we will reproduce this result by computing the three-point function (on the sphere) of the dual worldsheet vertex operators of string theory on $AdS_3 \times S^3 \times T^4$.

3. Superstring theory on $AdS_3 \times S^3 \times T^4$

The AdS/CFT correspondence relates the above 2-dimensional conformal field theory with string theory on $AdS_3 \times S^3 \times T^4$. This is the near-horizon geometry of the D1-D5 system. By S-duality we can relate this to the configuration of fundamental strings and NS5 branes. The latter system then has a description in terms of a WZW model. More precisely, the relevant world-sheet theory is the product of an $\mathcal{N} = 1$ WZW model on H_3^+, an $\mathcal{N} = 1$ WZW model on $S^3 \cong \text{SU}(2)$ and an $\mathcal{N} = 1$ $\text{U}(1)^4$ free superconformal field theory.

This WZW model has the affine world-sheet symmetry $\widehat{sl}(2)_k \times \widehat{su}(2)_{k'} \times u(1)^4$. In the following we shall (as is usual for $\mathcal{N} = 1$ WZW models) decouple the fermions from

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the currents; the resulting bosonic currents (that then commute with the fermions) are J^a for SL(2), and K^a for SU(2). We denote the free fermions that come from the SL(2) part by ψ^a, while those from the SU(2) part are χ^a. (In either case, a takes the three values $a = (+, 0, -)$.) Finally the $u(1)^4$ symmetry is described in terms of free bosons as $i\partial Y^i$, and the corresponding free fermions are λ^i $(i = 1, 2, 3, 4)$.

Criticality of the fermionic string on $AdS_3 \times S^3$ requires the identification of the levels k and k' [6], $k = k'$.[3] Furthermore, the $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ global R-symmetry of the boundary conformal field theory corresponds to the isometry $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ of the three-sphere S^3. In the SU(2) WZW model this symmetry is identified with the horizontal subalgebras generated by the K^a_0 and $\bar{K}^a_0$ of the affine $\widehat{su}(2)_{k'}$ symmetry. The levels $k = k'$ are to be identified with the number N_5 of NS5-branes,

$$k = k' = N_5 \ . \tag{3.1}$$

Finally, it is common lore in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence to interpret the continuous SL(2) representation labels $(x, \bar{x})$ (that will be introduced momentarily) with the complex coordinates of the boundary conformal field theory [10].

For the comparison of worldsheet and boundary CFT correlation functions, we also have to identify the point in the moduli space of the D1-D5 system (or better the S-dual F1-NS5 system) at which string theory on $AdS_3 \times S^3 \times T^4$ is dual to the symmetric orbifold theory. In [7] it was argued that the symmetric orbifold corresponds to the point $N_1 = N$, $N_5 = 1$, where N_1 is the number of fundamental strings and N_5 the number of NS5-branes. At this point the AdS_3 radius $R_{\mathrm{AdS}} = l_s (g_6^2 N_1 N_5)^{\frac{1}{4}} = l_s \sqrt{N_5}$ (in the F1-NS5 system) is of order of the string scale ($g_6^2 = N_5/N_1$ is the six-dimensional string coupling), and supergravity is not a good approximation any more. We must therefore consider the full worldsheet theory.

On the other hand, we will only be able to calculate the world-sheet correlators on the sphere, whereas the full string theory amplitude also involves the contribution from arbitrary genus Riemann surfaces. We therefore need to work in the limit where g_s is small; since we have [6, 11]

$$g_s^2 = \frac{N_5}{N_1} \,\mathrm{Vol}(T^4) \,, \tag{3.2}$$

g_s is small if $\mathrm{Vol}(T^4) N_5 \ll N_1$. By T-duality arguments [6], the volume can be chosen as $\mathrm{Vol}(T^4) \geq 1$. At the point $N_1 = N$, $N_5 = 1$ and fixed volume $\mathrm{Vol}(T^4)$, the worldsheet theory is weakly coupled if N is large such that $\mathrm{Vol}(T^4) \ll N$. Note that the AdS space is still strongly curved in the large N limit and the WZW model is the only reliable description.

Unfortunately, the worldsheet model is not properly defined for $(N_1, N_5) = (N, 1)$, since the bosonic level $k'_{\mathrm{bos}} = N_5 - 2 = -1$ is negative at this point. We therefore choose

[3] We shall use the convention that k and k' denote the levels of the supersymmetric $\mathcal{N} = 1$ theory; the levels of the bosonic affine symmetries are then shifted by the dual Coxeter numbers, $k_{\mathrm{bos}} = k_{\mathrm{susy}} + 2$ and $k'_{\mathrm{bos}} = k'_{\mathrm{susy}} - 2$.

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N_1 and N_5 as $(N_1, N_5) = (N/N_5, N_5)$ with $N_5 > 1$ but fixed, such that g_s remains small for large N. Even though we will compute the correlators at a point in the moduli space different from the orbifold point, we will see below that the worldsheet correlators do not depend on the actual factorisation of $N = N_1 N_5$, but only on N. It is therefore natural to expect that the results will agree with those expected from the orbifold theory, at least at large N.

3.1 The chiral primaries in the worldsheet theory

After these preliminary discussions we now need to study the fields of this world-sheet theory. First we describe the left-moving fields. The highest weight states of the SL(2) and SU(2) WZW model are denoted by $\Phi_{j,m}$ and $\Phi'_{j',m'}$, respectively; their conformal dimensions are

$$\Delta_j = -\frac{j(j-1)}{k} \quad \text{and} \quad \Delta'_{j'} = \frac{j'(j'+1)}{k'} \ . \tag{3.3}$$

The OPE's of the bosonic SL(2) currents J^a with the primary fields $\Phi_{j,m}$ are given as

$$\begin{aligned} J^{\pm}(z)\Phi_{j,m}(w) &= \frac{m \mp (j-1)}{z-w}\Phi_{j,m\pm1}(w) + :J^{\pm}\Phi_{j,m}(w): + \dots\,, \\ J^{0}(z)\Phi_{j,m}(w) &= \frac{m}{z-w}\Phi_{j,m}(w) + :J^{0}\Phi_{j,m}(w): + \dots\,, \end{aligned} \tag{3.4}$$

where $j \geq 1$.[4] The OPEs of the K^a-currents with the primary fields $\Phi'_{j',m'}$ are similar, but we do not need them in the following; more details about the SL(2) and SU(2) primary fields and their correlators can be found in appendix A.

To construct the 'chiral primaries' of the worldsheet theory on $AdS_3 \times S^3 \times T^4$ that correspond to the chiral primaries of the boundary conformal field theory described in section 2, we now need to tensor these left-moving fields with corresponding right-moving fields. Furthermore, we have to make sure that the fields survive the GSO-projection and have the correct ghost number. In the following we shall only consider the NS sector. The left-moving fields of interest are then [11]

$$\mathcal{W}_{j',m',m} = e^{-\phi}(\psi\Phi)_{j-1,m}\Phi'_{j',m'}\,, \qquad (j' = 0, \tfrac{1}{2}, \dots, \tfrac{k'}{2} - 1) \tag{3.5}$$

where

$$(\psi\Phi)_{j-1,m} = \psi^0\Phi_{j,m} - \frac{1}{2}\psi^+\Phi_{j,m-1} - \frac{1}{2}\psi^-\Phi_{j,m+1}\ . \tag{3.6}$$

The bosonised superghost fields $e^{-\phi}$ ensure that the operator $\mathcal{W}_{j',m',m}$ has the correct ghost number -1. Furthermore, in order to guarantee that $\mathcal{W}_{j',m',m}$ has also the correct conformal weight we need to set

$$j = j' + 1\,, \tag{3.7}$$

[4] We are using here the same conventions as in KLL [11] except that $j_{\rm KLL} = j - 1$. However these representations are not of the type discussed by [8] in their analysis of the bosonic WZW model corresponding to SL(2).

which justifies dropping the j label in the definition of $\mathcal{W}_{j',m',m}$. In fact, using that $k=k'$, we have

$$\Delta(\mathcal{W}_{j',m',m})=\Delta(e^{-\phi})+\Delta(\psi)-\frac{j(j-1)}{k}+\frac{j'(j'+1)}{k'}=1\,, \tag{3.8}$$

and since $\Delta(e^{-\phi})=1/2$, the conformal dimension of the matter part is indeed $\Delta=\frac{1}{2}$, as required. Since $k'=N_5$, only N_5-1 of the N n-cycle twist operators in the orbifold theory have a dual worldsheet operator of the type (3.5); in the following we shall only consider those. It is generally believed that the remaining twist operators are dual to worldsheet operators involving spectrally flowed $SL(2)$ representations [23, 24].

Introducing the compact notation

$$\psi^a=(\psi^+,\psi^0,\psi^-)\,,\qquad b_a=(-\tfrac{1}{2},1,-\tfrac{1}{2})\qquad a=(+,0,-)\,. \tag{3.9}$$

the operator $\mathcal{W}_{j',m',m}$ can be written as

$$\mathcal{W}_{j',m',m}=e^{-\phi}b_a\psi^a\Phi_{j,m-a}\Phi'_{j',m'}\ . \tag{3.10}$$

The analysis for the right-movers is identical, and the full local fields are then [11] (see also [6, 25])

$$V^{(1)}_{j',m',\bar{m}',m,\bar{m}}(z,\bar{z})=\mathcal{W}_{j',m',m}(z)\,\bar{\mathcal{W}}_{j',\bar{m}',\bar{m}}(\bar{z})\ . \tag{3.11}$$

The chiral primary $V^{(1)}_{j',m',\bar{m}',m,\bar{m}}$ is the 'Fourier transform' of the operator

$$V^{(1)}_{j',m',\bar{m}'}(z,\bar{z};x,\bar{x})=\sum_{m,\bar{m}}V^{(1)}_{j',m',\bar{m}',m,\bar{m}}(z,\bar{z})\,x^{-m-j'-1}\bar{x}^{-\bar{m}-j'-1}\,, \tag{3.12}$$

which depends both on the worldsheet coordinates $(z,\bar{z})$ as well as on the representation labels $(x,\bar{x})$. In addition to the worldsheet conformal dimension $(\Delta,\bar{\Delta})=(1,1)$, the operators $V^{(1)}_{j',m',\bar{m}'}(z,\bar{z};x,\bar{x})$ have also spacetime scaling dimensions $(h,\bar{h})=(j',j')$ [11]. Since the variables x and $\bar{x}$ are to be identified with the complex coordinates of the 2d conformal field theory on the boundary, these dimensions predict the scaling of the two-point function of $V^{(1)}_{j',m',\bar{m}'}(z,\bar{z};x,\bar{x})$ as $|z_{12}|^{-4\Delta}$ and $|x_{12}|^{-4h}$.

For the analysis of the 3-point functions we also need the corresponding operators with ghost number zero

$$V^{(0)}_{j',m',\bar{m}',m,\bar{m}}(z,\bar{z})=\mathcal{W}^0_{j',m',m}(z)\,\bar{\mathcal{W}}^0_{j',\bar{m}',\bar{m}}(\bar{z})\,, \tag{3.13}$$

that are obtained from $V^{(1)}_{j',m',\bar{m}',m,\bar{m}}$ by acting with the picture changing operator Γ_{+1}. The only non-vanishing contribution comes from $e^{\phi}G_{-\frac{1}{2}}$ in Γ_{+1}, and hence one finds

$$\mathcal{W}^0_{j',m',m}=e^{\phi}G_{-\frac{1}{2}}\mathcal{W}_{j',m',m}\,, \tag{3.14}$$

where the global $\mathcal{N}=1$ superconformal generator $G_{-\frac{1}{2}}=\frac{1}{2\pi i}\oint G(z)$ is given by the supercurrent $G(z)$

$$G(z)=\frac{2}{k}\left(\psi^aJ_a-\frac{i}{3k}f_{abc}\psi^a\psi^b\psi^c\right)+\frac{2}{k}\left(\chi^aK_a-\frac{i}{3k}f'_{abc}\chi^a\chi^b\chi^c\right)+\lambda^i\partial Y_i\ . \tag{3.15}$$

Using the SL(2) structure constants f_{abc}, we can write the SL(2) part of $G(z)$ as

$$G(z)\Big|_{\mathrm{SL}(2)} = \frac{1}{k}\left(\psi^+ J^- + \psi^- J^+ - 2\psi^0 J^0 + \frac{2}{k}\psi^+\psi^0\psi^-\right) . \tag{3.16}$$

After some algebra, we find, more explicitly,

$$\mathcal{W}^0_{j',m',m} = b_a : J^a \Phi_{j,m-a} : \Phi'_{j',m'} + \frac{2}{k} A_{ab} \psi^a \psi^b \Phi_{j,m-a-b} \Phi'_{j',m'} \equiv \mathcal{W}^{0,A}_{j',m',m} + \mathcal{W}^{0,B}_{j',m',m} , \tag{3.17}$$

where the matrix $A_{ab} = A^{(1)}_{ab} + A^{(2)}_{ab}$ is given by

$$A^{(1)}_{ab} = -d_a^{m-b} b_b , \qquad A^{(2)}_{ab} = \begin{pmatrix} 0 & b_- & 0 \\ 0 & 0 & b_+ \\ -\frac{1}{2}b_0 & 0 & 0 \end{pmatrix} , \qquad (a,b = +,0,-) \tag{3.18}$$

and

$$b_a = (-\tfrac{1}{2}, 1, -\tfrac{1}{2}) , \qquad d_a^m = (-\tfrac{1}{2}(j-1+m), m, \tfrac{1}{2}(j-1-m)) \qquad (a = +,0,-) . \tag{3.19}$$

For later convenience we denote the two terms in eq. (3.17) by $\mathcal{W}^{0,A}_{j',m',m}$ and $\mathcal{W}^{0,B}_{j',m',m}$, respectively. We note that in writing $\mathcal{W}^0_{j',m',m}$ we ignored terms coming the SU(2) part of $G(z)$, such as $\psi^a \chi^b \Phi_{j,m-a} \Phi'_{j',m'-b}$. For reasons discussed below, we do not need the precise form of these terms. (The U(1)-part of G acts anyway trivial since $\mathcal{W}$ is in the ground state with respect to the U(1) excitations.)

3.2 Correlators in the worldsheet theory

It is suggested in [11] that the operators $V^{(1)}_{j',m',\bar{m}',m,\bar{m}}$ and $V^{(0)}_{j',m',\bar{m}',m,\bar{m}}$ of the world-sheet theory correspond to the chiral primary operators $V_{j',m',\bar{m}'}$ of the boundary conformal field theory. We now want to check this identification by comparing their correlation functions.

3.2.1 Worldsheet two-point function

Let us begin with the two-point function of the chiral primary field $V^{(1)}_{j',m',\bar{m}',m,\bar{m}}(z,\bar{z})$,

$$G_2 = g_s^{-2} \langle V^{(1)}_{j',m',\bar{m}',m,\bar{m}}(z_1,\bar{z}_1)\, V^{(1)}_{j',-m',-\bar{m}',-m,-\bar{m}}(z_2,\bar{z}_2)\rangle_{S^2} , \tag{3.20}$$

which will be evaluated on the sphere. Substituting the explicit expression of $V^{(1)}_{j',m',\bar{m}',m,\bar{m}}$, eq. (3.11), into G_2, we get

$$\begin{aligned} G_2 = g_s^{-2} b_{a_1} b_{\bar{a}_1} b_{a_2} b_{\bar{a}_2} \langle \psi^{a_1}\psi^{a_2}\rangle \langle \bar{\psi}^{\bar{a}_1}\bar{\psi}^{\bar{a}_2}\rangle |z_{12}|^{-2} \\ \times \langle \Phi_{j,m-a_1,\bar{m}-\bar{a}_1} \Phi_{j,-m-a_2,-\bar{m}-\bar{a}_2}\rangle \langle \Phi'_{j',m',\bar{m}'} \Phi'_{j',-m',-\bar{m}'}\rangle , \end{aligned} \tag{3.21}$$

where the summation over $a_1, \bar{a}_1, a_2, \bar{a}_2$ is understood. The factor $|z_{12}|^{-2}$ comes from the propagator of the ghosts. Using the form of the 2-point function of the SU(2) chiral primary fields $\Phi'_{j',m',\bar{m}'}$ given in eq. (A.16), and that of the fermions ψ^a,

$$\langle \psi^{a_1}(z_1)\psi^{a_2}(z_2)\rangle = \frac{k\,\eta^{a_1,a_2}}{2\,z_{12}} \qquad (a_1,a_2 = 0,\pm 1) , \tag{3.22}$$

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where $\eta^{+-} = 2$ and $\eta^{00} = -1$, the two-point function G_2 simplifies to

$$G_2 = g_s^{-2} \frac{k^2}{4} \frac{b_{a_1} b_{\bar{a}_1} b_{a_2} b_{\bar{a}_2}}{|z_{12}|^{4\Delta'_{j'}+4}} \eta^{a_1 a_2} \eta^{\bar{a}_1 \bar{a}_2} \langle \Phi_{j,m-a_1,\bar{m}-\bar{a}_1} \Phi_{j,-m-a_2,-\bar{m}-\bar{a}_2} \rangle \ . \tag{3.23}$$

The only nonvanishing terms in G_2 are those for which $a_1 = -a_2$ and $\bar{a}_1 = -\bar{a}_2$.

Next we employ the formula for the two-point function of the SL(2) primaries in m-space,

$$\langle \Phi_{j,m,\bar{m}} \Phi_{j,-m,-\bar{m}} \rangle = \frac{\pi B(j) \delta(0)}{|z_{12}|^{4\Delta_j}} \frac{\Gamma(1-2j)\Gamma(j-m)\Gamma(j+\bar{m})}{\Gamma(2j)\Gamma(1-j-m)\Gamma(1+\bar{m}-j)} \,, \tag{3.24}$$

with $B(j)$ as defined in eq. (A.5). Then we sum over a_1 and $\bar{a}_1$ (nine terms) and obtain

$$G_2 = g_s^{-2} \frac{k^2}{16} \frac{\pi B(j) \delta(0)}{|z_{12}|^{4(\Delta_j + \Delta'_{j'}+1)}} \frac{\Gamma(1-2j')\Gamma(j'-m)\Gamma(j'+\bar{m})}{\Gamma(2j')\Gamma(1-j'-m)\Gamma(1+\bar{m}-j')} \ . \tag{3.25}$$

It is interesting to observe that in comparison with the two-point function (3.24), the arguments in the gamma functions in G_2 are shifted from j to $j' = j-1$. This shows that in the definition of the chiral primary the action of ψ^a on the SL(2) operators $\Phi_{j,m,\bar{m}}$ shifts the x-dependence of the two-point function from $|x_{12}|^{-4j}$ to $|x_{12}|^{-4(j-1)}$. Transforming G_2 back to coordinate space, we obtain

$$G_2 = g_s^{-2} \frac{k^2}{16} \frac{B(j) \delta(0)}{|x_{12}|^{4(j-1)}} \frac{1}{|z_{12}|^4} \tag{3.26}$$

which has the expected scaling behavior, $|z_{12}|^{-4\Delta}$ and $|x_{12}|^{-4h}$ ($\Delta = 1$, $h = j'$).

In order to obtain the two-point function of the corresponding n-cycle twist operators of the boundary conformal field theory from this, we now follow the prescription given in [8]. We find

$$\begin{aligned} &\langle V_{j',m',\bar{m}'}(x_1,\bar{x}_1)\, V_{j',-m',-\bar{m}'}(x_2,\bar{x}_2) \rangle_{\text{bCFT}} \\ &\quad = \frac{1}{V_{\text{conf}}} g_s^{-2} \, \langle V^{(1)}_{j',m',\bar{m}'}(z_1 = \bar{z}_1 = 1; x_1, \bar{x}_1)\, V^{(1)}_{j',-m',-\bar{m}'}(z_2 = \bar{z}_2 = 0; x_2, \bar{x}_2) \rangle_{S^2} \\ &\quad = g_s^{-2} \, \frac{k^2}{16} \, \frac{B(j)}{|x_{12}|^{4(j-1)}} \, \frac{\delta(0)}{V_{\text{conf}}} \\ &\quad = (2j-1)\, g_s^{-2} \, \frac{k^2}{16} \, \frac{B(j)}{|x_{12}|^{4j'}} \ . \end{aligned} \tag{3.27}$$

Here $V_{\text{conf}} = \int d^2z |z|^{-2}$ is the volume of the conformal group on the sphere (the Möbius group); this factor cancels the divergence coming from the delta function $\delta(0)$ in eq. (3.26) up to a j-dependent factor, which is $2j-1$ [8].

Now we can compare this result to the boundary two-point function (2.8) found in the symmetric orbifold theory. In particular, this now fixes the relative normalisation between the worldsheet vertex operators $V^{(1)}_{j',m',\bar{m}'}(z,\bar{z};x,\bar{x})$ and the conformal field theory operators $V_{j',m',\bar{m}'}(x,\bar{x})$

$$V^{(1)}_{j',m',\bar{m}'}(z,\bar{z};x,\bar{x}) \Longleftrightarrow a(j)\, V_{j',m',\bar{m}'}(x,\bar{x}) \,, \tag{3.28}$$

where

$$a(j) = \frac{k}{g_s 4}\sqrt{(2j-1)B(j)}\ . \tag{3.29}$$

3.2.2 Worldsheet three-point function

We now consider the three-point function

$$G_3 = g_s^2 \left\langle V^{(1)}_{j_1',m_1',\bar m_1',m_1,\bar m_1}(z_1,\bar z_1)\, V^{(1)}_{j_2',m_2',\bar m_2',m_2,\bar m_2}(z_2,\bar z_2)\, V^{(0)}_{j_3',m_3',\bar m_3',m_3,\bar m_3}(z_3,\bar z_3)\right\rangle_{S^2}\ , \tag{3.30}$$

where $V^{(0)}_{j',m',\bar m',m,\bar m}$ is the descendant of the chiral primary $V^{(1)}_{j',m',\bar m',m,\bar m}$. Since $V^{(1)}$ has ghost number -1, while $V^{(0)}$ has ghost number zero, the total ghost number of G_3 is then -2, as required on the sphere. The explicit calculation can now be divided into four parts

$$G_3 = G_3^{AA} + G_3^{AB} + G_3^{BA} + G_3^{BB}\ , \tag{3.31}$$

where

$$\begin{aligned}
G_3^{AA} &= g_s^{-2}\left\langle V^{(1)}_{j_1',m_1',\bar m_1',m_1,\bar m_1}(z_1,\bar z_1)\, V^{(1)}_{j_2',m_2',\bar m_2',m_2,\bar m_2}(z_2,\bar z_2)\, \mathcal{W}^{0,A}_{j_3',m_3',m_3}(z_3)\, \bar{\mathcal{W}}^{0,A}_{j_3',\bar m_3',\bar m_3}(\bar z_3)\right\rangle\\
G_3^{AB} &= g_s^{-2}\left\langle V^{(1)}_{j_1',m_1',\bar m_1',m_1,\bar m_1}(z_1,\bar z_1)\, V^{(1)}_{j_2',m_2',\bar m_2',m_2,\bar m_2}(z_2,\bar z_2)\, \mathcal{W}^{0,A}_{j_3',m_3',m_3}(z_3)\, \bar{\mathcal{W}}^{0,B}_{j_3',\bar m_3',\bar m_3}(\bar z_3)\right\rangle\\
G_3^{BA} &= g_s^{-2}\left\langle V^{(1)}_{j_1',m_1',\bar m_1',m_1,\bar m_1}(z_1,\bar z_1)\, V^{(1)}_{j_2',m_2',\bar m_2',m_2,\bar m_2}(z_2,\bar z_2)\, \mathcal{W}^{0,B}_{j_3',m_3',m_3}(z_3)\, \bar{\mathcal{W}}^{0,A}_{j_3',\bar m_3',\bar m_3}(\bar z_3)\right\rangle\\
G_3^{BB} &= g_s^{-2}\left\langle V^{(1)}_{j_1',m_1',\bar m_1',m_1,\bar m_1}(z_1,\bar z_1)\, V^{(1)}_{j_2',m_2',\bar m_2',m_2,\bar m_2}(z_2,\bar z_2)\, \mathcal{W}^{0,B}_{j_3',m_3',m_3}(z_3)\, \bar{\mathcal{W}}^{0,B}_{j_3',\bar m_3',\bar m_3}(\bar z_3)\right\rangle\ ,
\end{aligned}$$

and $\mathcal{W}^{0,A}$ and $\mathcal{W}^{0,B}$ are defined as in eq. (3.17).

We begin with the correlator G_3^{BB}. Substituting the explicit expressions for the vertex operators, we get

$$\begin{aligned}
G_3^{BB} = g_s^{-2}\frac{4}{k^2} & b_{a_1} b_{a_2} A_{a_3 b_3} b_{\bar a_1} b_{\bar a_2} A_{\bar a_3 \bar b_3} \langle \psi^{a_1}\psi^{a_2}\psi^{a_3}\psi^{b_3}\rangle \langle \bar\psi^{\bar a_1}\bar\psi^{\bar a_2}\bar\psi^{\bar a_3}\bar\psi^{\bar b_3}\rangle\\
&\times \langle \Phi_{j_1,m_1-a_1,\bar m_1-\bar a_1}\Phi_{j_2,m_2-a_2,\bar m_2-\bar a_2}\Phi_{j_3,m_3-a_3-b_3,\bar m_3-\bar a_3-\bar b_3}\rangle\\
&\times \langle \Phi'_{j_1',m_1',\bar m_1'}\Phi'_{j_2',m_2',\bar m_2'}\Phi'_{j_3',m_3',\bar m_3'}\rangle |z_{12}|^{-2}\ ,
\end{aligned} \tag{3.32}$$

where summation over $a_i, b_3, \bar a_i, \bar b_3$ $(i=1,2,3)$ is understood. Here we have already omitted terms in the operator $V^{(0)}_{j',m',\bar m',m,\bar m}$ which involve the spinor χ^a (*cf.* the discussion at the end of section 3.1). Since the SU(2) spinor χ^a cannot be contracted with the SL(2) spinor ψ^a, such terms do not contribute to the three-point function. The factor $|z_{12}|^{-2}$ is as before the contribution from the ghosts.

Let us consider in detail the three-point functions of the primary fields ψ^a, $\Phi_{j,m,\bar m}$ and $\Phi'_{j',m',\bar m'}$. The three-point function involving the fermions ψ^a in G_3^{BB} is evaluated using Wick contraction and the fermion propagator (3.22)

$$\langle \psi^{a_1}(z_1)\psi^{a_2}(z_2)\psi^{a_3}(z_3)\psi^{b_3}(z_3)\rangle = k^2\frac{\eta^{a_1[b_3}\eta^{a_3]a_2}}{z_{13}z_{23}}\ . \tag{3.33}$$

This is non-vanishing only if

$$(a_1 = -b_3\,,\; a_2 = -a_3\,,\; a_3 \neq b_3) \qquad \text{or} \qquad (a_1 = -a_3\,,\; a_2 = -b_3\,,\; a_3 \neq b_3)\,. \tag{3.34}$$

These constraints imply the relation $a_1 + a_2 + a_3 + b_3 = 0$.

The three-point functions of the SL(2) and SU(2) primary fields $\Phi_{j,m,\bar{m}}$ and $\Phi'_{j',m',\bar{m}'}$ are given by

$$\begin{aligned}\langle \Phi_{j_1,m_1,\bar{m}_1}(z_1,\bar{z}_1)\, \Phi_{j_2,m_2,\bar{m}_2}(z_2,\bar{z}_2)\, \Phi_{j_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle \\ = \delta^2(m_1+m_2+m_3)\, W(j_i;m_i)\, C_{j_1,j_2,j_3} \prod_{i<j} \frac{1}{|z_{ij}|^{2\Delta_{ij}}}\end{aligned} \tag{3.35}$$

$$\begin{aligned}\langle \Phi'_{j'_1,m'_1,\bar{m}'_1}(z_1,\bar{z}_1)\, \Phi'_{j'_2,m'_2,\bar{m}'_2}(z_2,\bar{z}_2)\, \Phi'_{j'_3,m'_3,\bar{m}'_3}(z_3,\bar{z}_3)\rangle \\ = \delta^2(m'_1+m'_2+m'_3)\, \hat{W}(j'_i;m'_i)\, C'_{j'_1,j'_2,j'_3} \prod_{i<j} \frac{1}{|z_{ij}|^{2\Delta'_{ij}}}\,,\end{aligned} \tag{3.36}$$

where C_{j_1,j_2,j_3} and $C'_{j'_1,j'_2,j'_3}$ are the SL(2) and SU(2) structure constants, respectively, that are explicitly given in eqs. (A.7) and (A.18) of appendix A. We have used the conventions $x_{12} = x_1 - x_2$, $z_{12} = z_1 - z_2$, $\Delta_{12} = \Delta_{j_1} + \Delta_{j_2} - \Delta_{j_3}$, $j_{12} = j_1 + j_2 - j_3$, etc. The δ^2 symbol means that the three-point function (3.35) is only nonvanishing for $m_1 + m_2 + m_3 = \bar{m}_1 + \bar{m}_2 + \bar{m}_3 = 0$, and similarly for (3.36).

The function $W(j_i,m_i)$ is the 'Fourier transform' of the factor $\prod_{i<j} |x_{ij}|^{-2j_{ij}}$, i.e. it reflects the dependence on the representation label x. This function is given by the integral

$$\begin{aligned}W(j_i;m_i) = \int d^2x_2\, d^2x_3\, x_2^{j_2+m_2-1} \bar{x}_2^{j_2+\bar{m}_2-1} |1-x_2|^{-2j_{12}} \\ \times x_3^{j_3+m_3-1} \bar{x}_3^{j_3+\bar{m}_3-1} |1-x_3|^{-2j_{13}} |x_2-x_3|^{-2j_{23}}\,.\end{aligned} \tag{3.37}$$

An explicit expression for this integral has been found by Satoh [26],[5]

$$\begin{aligned}W(j_i;m_i) &= (-)^w \frac{\pi^2\gamma(-\tilde{N})\gamma(2j'_3+1)}{\gamma(1+j'_{31})\gamma(1+j'_{32})} \frac{\Gamma(1+j'_2-m_2)\Gamma(1+j'_2-\bar{m}_2)}{\Gamma(1+j'_2-m_2-n_3)\Gamma(1+j'_2-\bar{m}_2-\bar{n}_3)} \\ &\times \prod_{a=1,2} \frac{\Gamma(1+j'_a+m_a)}{\Gamma(-j'_a-\bar{m}_a)} F\begin{bmatrix} -n_3, -j'_{31}, 1+j'_{12} \\ -2j'_3, 1+j'_2-m_2-n_3 \end{bmatrix} F\begin{bmatrix} -\bar{n}_3, -j'_{31}, 1+j'_{12} \\ -2j'_3, 1+j'_2-\bar{m}_2-\bar{n}_3 \end{bmatrix}\end{aligned} \tag{3.38}$$

with $w = m_2 - \bar{m}_2 + \bar{n}_3$ and $j'_i = j_i - 1$. Furthermore, $m_3 = -j'_3 + n_3$ with $n_3 \geq 0$, and $\bar{m}_3 = -j'_3 + \bar{n}_3$ with $\bar{n}_3 \geq 0$ (see appendix A). Finally, $F[a,b,c;e,f]$ is the hypergeometric function ${}_3F_2(a,b,c;e,f;1)$ and $\tilde{N} = j'_1 + j'_2 + j'_3 + 1$.

We want to compare this 3-point function with the 3-point function of the boundary conformal field theory (2.6). There we restricted ourselves to the case (2.6) with $d = j'_{12} \geq 0$. For these special values also the function $\hat{W}(j'_i;m'_i)$ simplifies, and we find

$$\hat{W}(j'_i;m'_i) = \frac{\Gamma(j'_{13}+1)\Gamma(j'_{12}+1)}{\Gamma(2j'_1+1)}\,. \tag{3.39}$$

[5] In Satoh's analysis this integral appears in a slightly different context as he uses different conventions for the SL(2) representations.

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This is shown in appendix A, using the SU(2) OPE coefficients [17] for the case described by (2.6). Substituting eqs. (3.33), (3.35) and (3.36) in G_3^{BB}, we obtain

$$\begin{aligned} G_3^{BB} &= g_s^{-2} k^2\, b_{a_1} b_{a_2} A_{a_3 b_3} b_{\bar{a}_1} b_{\bar{a}_2} A_{\bar{a}_3 \bar{b}_3}\, \eta^{a_1[b_3}\eta^{a_3]a_2}\eta^{\bar{a}_1[\bar{b}_3}\eta^{\bar{a}_3]\bar{a}_2}\, C_{j_1,j_2,j_3} C'_{j'_1,j'_2,j'_3} \delta^2(\textstyle\sum_i m_i) \\ &\quad \times W(j_i; m_1 - a_1, m_2 - a_2, m_3 - a_3 - b_3) \hat{W}(j'_i; m'_i) \prod_{i<j} \frac{1}{|z_{ij}|^{2(\Delta_{ij}+\Delta'_{ij}+1)}} \,. \end{aligned} \tag{3.40}$$

We will now perform the sum over the eight indices $a_i, b_3, \bar{a}_i, \bar{b}_3$ ($i = 1, 2, 3$). We expect that after taking the sum, the function $W(j_i; \ldots)$ in G_3^{BB} is shifted to $W(j_i - 1; m_i)$, which is the 'Fourier transform' of $\prod_{i<j} |x_{ij}|^{-2j_{ij}+2} = \prod_{i<j} |x_{ij}|^{-2j'_{ij}}$. G_3^{BB} would then have the same dependence on the coordinates x_i ($i = 1, 2, 3$) as the boundary three-point function (2.9). In order to check this, we consider the special case where the SL(2) quantum numbers are[6]

$$\begin{aligned} m_1 &= \bar{m}_1 = j'_1 - d\,, \\ m_2 &= \bar{m}_2 = j'_2 - 1\,, \\ m_3 &= \bar{m}_3 = -j'_3 + 1 = -(j'_1 + j'_2 - d) + 1\,. \end{aligned} \tag{3.41}$$

The condition that $n_3 \geq 0$ then translates into

$$n_3 = 1 - a_3 - b_3 \geq 0\,. \tag{3.42}$$

The constraints (3.34) and (3.42) (and similar constraints for the bared indices) imply that there are 144 nonvanishing terms in G_3^{BB} which we sum up using computer algebra. We then obtain

$$G_3^{BB} = g_s^{-2} g^{BB}(j'_i)\, C_{j_1,j_2,j_3}\, C'_{j'_1,j'_2,j'_3}\, W(j_i - 1; m_i)\, \hat{W}(j'_i; m'_i) \prod_{i<j} \frac{1}{|z_{ij}|^{2(\Delta_{ij}+\Delta'_{ij}+1)}}\,, \tag{3.43}$$

where the function $g^{BB}(j'_i)$ turns out to be

$$g^{BB}(j'_i) = \frac{k^2}{16}(2j'_3)^2\,. \tag{3.44}$$

In particular, this expression shows the desired shift in the j_i dependence of W.

We proceed similarly for the remaining terms G_3^{AA}, G_3^{AB} and G_3^{BA}. As shown in appendix B, these terms have the same structure as G_3^{BB}, but with $g^{BB}(j'_i)$ replaced by

$$g^{AA}(j'_i) = \frac{k^2}{16}\,(1 + j'_1 + j'_2 - j'_3)^2\,, \tag{3.45}$$

$$g^{AB}(j'_i) = g^{BA}(j'_i) = \frac{k^2}{16}\,(1 + j'_1 + j'_2 - j'_3) 2j'_3\,. \tag{3.46}$$

After adding up the four terms, we obtain

$$G_3 = g_s^2\, g(j'_i)\, C_{j_1,j_2,j_3}\, C'_{j'_1,j'_2,j'_3}\, W(j_i - 1; m_i)\, \hat{W}(j'_i; m'_i) \prod_{i<j} \frac{1}{|z_{ij}|^{2(\Delta_{ij}+\Delta'_{ij}+1)}}\,, \tag{3.47}$$

[6] Because of the SL(2) covariance, this result should then hold for arbitrary values of m_i, $\bar{m}_i$ ($i = 1, 2, 3$).

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with

$$g(j_i') = g^{AA}(j_i') + g^{AB}(j_i') + g^{BA}(j_i') + g^{BB}(j_i') = \frac{k^2}{16}\,(1 + j_1' + j_2' + j_3')^2\,. \tag{3.48}$$

The product of the SL(2) and SU(2) structure constants C_{j_1,j_2,j_3} and $C'_{j_1',j_2',j_3'}$ in G_3 can now be simplified using the identity

$$C_{j_1,j_2,j_3} C'_{j_1',j_2',j_3'} = \sqrt{B(j_1)B(j_2)B(j_3)}\,, \tag{3.49}$$

which is shown in appendix C. This finally allows us to write G_3 as

$$G_3 = g_s^2\,\frac{k^2}{16}\,(j_1' + j_2' + j_3' + 1)^2\, W(j_i - 1; m_i)\, \hat{W}(j_i'; m_i') \prod_i B(j_i)^{1/2} \prod_{i<j} \frac{1}{|z_{ij}|^2}\,. \tag{3.50}$$

Before comparing this to the dual boundary conformal field theory answer let us pause to comment on the identity (3.49) which is somewhat striking: it states that, for canonically normalised fields, the operator algebra coefficients of $\mathrm{SL}(2)_k$ are inverse to those of $\mathrm{SU}(2)_{k'}$ (for $k = k'$)! The technical reason for this is that the functions $G(j)$ appearing in C_{j_1,j_2,j_3} and the functions $P(j')$ occurring in $C'_{j_1',j_2',j_3'}$ behave inversely to one another. More precisely, from the definition of the functions $G(j)$ and $P(j')$, we obtain the simple relation

$$G(-j) = \frac{G(-1)}{P(j')}\,, \tag{3.51}$$

where $G(-1)$ is a regular function, see appendix C for details. In the product $C \cdot C'$ the poles of the structure constants C_{j_1,j_2,j_3} therefore cancel precisely against the zeros of $C'_{j_1',j_2',j_3'}$.

Returning to the question of the AdS/CFT correspondence, we need to transform G_3 back into x-space in order to compare it with the boundary three-point function (2.9). This is now trivial since we can use the Satoh formula again, except that now j_i has been shifted to $j_i' = j_i - 1$, as already discussed above. In particular, this therefore reproduces the same x-dependence as in (2.9). It thus remains to check that also the overall factor of the three-point functions agree. From the analysis of the two-point function we have deduced how the fields have to be rescaled, see eq. (3.28). Taking this into account, we then obtain from G_3 the rescaled function

$$\widehat{G}_3 = \frac{g_s 4}{k}\,(j_1' + j_2' + j_3' + 1)^2\, \frac{\Gamma(j_{13}' + 1)\Gamma(j_{12}' + 1)}{\Gamma(2j_1' + 1)} \prod_i (2j_i' + 1)^{-1/2} \prod_{i<j} \frac{1}{|x_{ij}|^{2j_{ij}'}}\,, \tag{3.52}$$

where we have also performed the integral over the world-sheet coordinates, which in this case just cancels the volume of the Möbius group. Since $k = N_5$, $\widehat{G}_3$ scales as $g_s/k \sim 1/\sqrt{N_1 N_5} = 1/\sqrt{N}$. This large N scaling behavior agrees then precisely with that of the boundary three-point function (2.9), see (2.13). This is a non-trivial consistency check since the functional dependence on the j_i' is quite complicated!

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4. Conclusions

In this paper we have compared correlation functions of chiral primary operators of the $AdS_3 \times S^3 \times T^4$ WZW model with the corresponding amplitudes in the boundary conformal field theory that can be defined as a symmetric orbifold. The comparison of the 2-point functions determines the relative normalisation of the chiral primary fields in the two descriptions. It is then a non-trivial consistency check to compare the coefficients of the 3-point functions. In the large N limit in which the sphere correlators (that we have calculated) dominate the string perturbation series, we have found beautiful agreement. We should note that the chiral primaries we have considered lie in short multiplets and are therefore protected by non-renormalisation theorems. It therefore makes sense to compare these correlation functions.

It would be interesting, though technically demanding, to repeat this analysis for the 4-point functions. This is the first example where in the world-sheet theory a non-trivial integral over the cross-ratio will have to be performed. It would be interesting to understand in detail how this will manifest itself in the dual boundary conformal field theory. For the case of the usual $AdS_5 \times S^5$ case, Gopakumar has suggested [9] that this integral becomes the Schwinger parametrisation of the propagator. This idea has recently been tested a little bit further [27, 28].

Note added: After completion of this paper the paper [30] appeared which contains some overlapping results.

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A. Correlators in $\mathrm{SL}(2)_k$ and $\mathrm{SU}(2)_{k'}$ WZW models

A.1 Two- and three-point functions in the $\mathrm{SL}(2)_k$ WZW model

The chiral primaries of the SL(2) WZW model are denoted by[7]

$$\Phi_{j,m,\bar{m}}(z,\bar{z}) = \Phi_{j,m}(z)\,\bar{\Phi}_{j,\bar{m}}(\bar{z}) \qquad \text{with} \qquad \Delta_j = \bar{\Delta}_j = -\frac{j(j-1)}{k-2}\,, \tag{A.1}$$

where k is the level of the affine Lie algebra. In the current context only half-integer j will be relevant (because of (3.7)). In this case, the OPEs (3.4) imply that the values of m and $\bar{m}$ run between $m = -(j-1), \ldots, (j-1)$.

[7] In this appendix we only deal with the bosonic currents; k and k' therefore refer to the bosonic levels.

The vertex operators $\Phi_{j,m,\bar{m}}(z,\bar{z})$ are the 'Fourier transforms' of the operators $\Phi_j(z,\bar{z};x,\bar{x})$

$$\Phi_{j,m,\bar{m}}(z,\bar{z}) = \int d^2x\, x^{j+m-1}\bar{x}^{j+\bar{m}-1}\Phi_j(z,\bar{z};x,\bar{x})\ . \tag{A.2}$$

The inverse transformation is

$$\Phi_j(z,\bar{z};x,\bar{x}) = \frac{1}{V_{\rm conf}}\sum_{m,\bar{m}}\Phi_{j,m,\bar{m}}(z,\bar{z})\, x^{-m-j}\bar{x}^{-\bar{m}-j}\,, \tag{A.3}$$

where $V_{\rm conf} = \int d^2x|x|^{-2}$.

The two- and three-point functions of $\Phi_j(z,\bar{z};x,\bar{x})$ were computed in [18 – 20]. The two-point function is given by[8]

$$\begin{aligned}&\langle\Phi_{j_1}(z_1,\bar{z}_1;x_1,\bar{x}_1)\Phi_{j_2}(z_2,\bar{z}_2;x_2,\bar{x}_2)\rangle \\ &\quad = \frac{1}{|z_{12}|^{4\Delta_{j_1}}}\left[\frac{1}{(2\pi)^2}\,\delta(x_{12})\,\delta(\bar{x}_{12})\,\delta(j_1+j_2-1) + \frac{B(j_1)}{|x_{12}|^{4j_1}}\delta(j_1-j_2)\right]\,,\end{aligned} \tag{A.4}$$

with coefficient

$$B(j) = \frac{1}{(2\pi)^2}\frac{k-2}{\pi}\frac{\nu^{1-2j}}{\gamma(\frac{2j-1}{k-2})} \qquad \text{and} \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,, \qquad \nu = \pi\frac{\Gamma(1-\frac{1}{k-2})}{\Gamma(1+\frac{1}{k-2})}\ . \tag{A.5}$$

The three-point function is

$$\langle\Phi_{j_1}(z_1,\bar{z}_1;x_1,\bar{x}_1)\,\Phi_{j_2}(z_2,\bar{z}_2;x_2,\bar{x}_2)\,\Phi_{j_3}(z_3,\bar{z}_3;x_3,\bar{x}_3)\rangle = C_{j_1j_2j_3}\prod_{i<j}\frac{1}{|x_{ij}|^{2j_{ij}}|z_{ij}|^{2\Delta_{ij}}}\,, \tag{A.6}$$

with $\Delta_{12} = \Delta_{j_1}+\Delta_{j_2}-\Delta_{j_3}$, $j_{12} = j_1+j_2-j_3$, etc. and coefficients

$$C_{j_1,j_2,j_3} = \frac{1}{(2\pi)^2}\,\frac{k-2}{2\pi^3}\,\frac{G(1-j_1-j_2-j_3)G(-j_{12})G(-j_{23})G(-j_{31})}{\nu^{j_1+j_2+j_3-2}G(-1)G(1-2j_1)G(1-2j_2)G(1-2j_3)}\,, \tag{A.7}$$

where

$$G(j) = (k-2)^{\frac{j(k-1-j)}{2(k-2)}}\,\Gamma_2(-j|1,k-2)\,\Gamma_2(k-1+j|1,k-2)\,, \tag{A.8}$$

and $\Gamma_2(x|1,\omega)$ is the Barnes double Gamma function. $G(j)$ has poles at $j = n+m(k-2)$ and $j = -n-1-(m+1)(k-2)$ with $n,m = 0,1,\ldots$. A discussion of these poles can be found in [8]. In C_{j_1,j_2,j_3} the poles $j_1+j_2+j_3 = n+k$, $n = 0,1,\ldots$ are excluded by the condition

$$j_1+j_2+j_3 \leq k-1\,. \tag{A.9}$$

[8]The SL(2) amplitudes were only determined up to an overall normalisation in [19]. In the following we shall use a different overall normalisation from [19], which seems to be more natural in the current context; this will become apparent in appendix C. We thank Jörg Teschner for explaining this to us.

The function $G(j)$ satisfies the recursion relation

$$G(j+1) = \gamma(-\tfrac{j+1}{k-2})\, G(j)\ . \tag{A.10}$$

In m-space the two- and three-point functions are given by

$$\langle \Phi_{j,m,\bar m}(z_1,\bar z_1)\Phi_{j,-m,-\bar m}(z_2,\bar z_2)\rangle = \frac{\pi B(j)\delta(0)}{|z_{12}|^{4\Delta_j}} \frac{\Gamma(1-2j)\Gamma(j-m)\Gamma(j+\bar m)}{\Gamma(2j)\Gamma(1-j-m)\Gamma(1+\bar m-j)} \tag{A.11}$$

and

$$\begin{aligned}\langle \Phi_{j_1,m_1,\bar m_1}(z_1,\bar z_1)&\Phi_{j_2,m_2,\bar m_2}(z_2,\bar z_2)\Phi_{j_3,m_3,\bar m_3}(z_3,\bar z_3)\rangle \\ &= \delta^2(m_1+m_2+m_3) W(j_a;m_a) C_{j_1,j_2,j_3} \prod_{i<j} \frac{1}{|z_{ij}|^{2\Delta_{ij}}}\,,\end{aligned} \tag{A.12}$$

with coefficients $B(j)$ and C_{j_1,j_2,j_3} as above.
An explicit expression for $W(j_i,m_i)$ can be found in [26]. Defining $j'_i = j_i - 1$ $(i=1,2,3)$, the integral (3.37) is identical to eq. (2.8) in [26] (upon cyclic permutation of the indices) which can written as

$$\begin{aligned} W(j_i;m_i) = (-)^{m_2-\bar m_2+\bar n_3} &\frac{\pi^2\gamma(-\tilde N)\gamma(2j'_3+1)}{\gamma(1+j'_{31})\gamma(1+j'_{32})} \frac{\Gamma(1+j'_2-m_2)\Gamma(1+j'_2-\bar m_2)}{\Gamma(1+j'_2-m_2-n_3)\Gamma(1+j'_2-\bar m_2-\bar n_3)} \\ \times \prod_{a=1,2} \frac{\Gamma(1+j'_a+m_a)}{\Gamma(-j'_a-\bar m_a)} &F\begin{bmatrix} -n_3, -j'_{31}, 1+j'_{12} \\ -2j'_3, 1+j'_2-m_2-n_3\end{bmatrix} F\begin{bmatrix} -\bar n_3, -j'_{31}, 1+j'_{12} \\ -2j'_3, 1+j'_2-\bar m_2-\bar n_3\end{bmatrix} \end{aligned} \tag{A.13}$$

where $F\begin{bmatrix} a,b,c \\ e,f\end{bmatrix}$ is the hypergeometric function ${}_3F_2(a,b,c;e,f;1)$. Here $\tilde N = j'_1+j'_2+j'_3+1$, $m_3 = -j'_3+n_3$ and $\bar m_3 = -j'_3+\bar n_3$ $(n_3,\bar n_3 = 0,1,\ldots)$; m_1, m_2 and $\bar m_1, \bar m_2$ are arbitrary.

A.2 Two- and three-point functions in the $\mathrm{SU}(2)_{k'}$ WZW model

The chiral primaries of the $\mathrm{SU}(2)_{k'}$ WZW model are denoted by

$$\Phi'_{j',m',\bar m'}(z,\bar z) = \Phi'_{j',m'}(z)\,\bar\Phi'_{j',\bar m'}(\bar z)\,, \tag{A.14}$$

and have conformal dimension

$$\Delta'_{j'} = \bar\Delta'_{j'} = \frac{j'(j'+1)}{k'+2}\,, \qquad 0 \le j' \le \frac{k'}{2}\,, \tag{A.15}$$

where j' is the SU(2) representation label and k' the level of the affine Lie algebra.

As for the case of SL(2) it is convenient to introduce instead of the m' variables continuous y variables (see for example [29]). In these conventions the two- and three-point functions of $\Phi'_{j'}(z,\bar z;y,\bar y)$ are then [16, 17, 29]

$$\langle \Phi'_{j'_1}(z_1,\bar z_1;y_1,\bar y_1)\Phi'_{j'_2}(z_2,\bar z_2;y_2,\bar y_2)\rangle = \delta_{j'_1,j'_2} \frac{|y_{12}|^{2j'_1}}{|z_{12}|^{4\Delta'_{j'_1}}}\,, \tag{A.16}$$

and

$$\langle \Phi'_{j'_1}(z_1,\bar z_1;y_1,\bar y_1)\Phi'_{j'_2}(z_2,\bar z_2;y_2,\bar y_2)\Phi'_{j'_3}(z_3,\bar z_3;y_3\bar y_3)\rangle = C'_{j'_1,j'_2,j'_3}\prod_{i<j}\frac{|y_{ij}|^{2j'_{ij}}}{|z_{ij}|^{2\Delta'_{ij}}}\,, \tag{A.17}$$

with $\Delta'_{12} = \Delta'_{j'_1} + \Delta'_{j'_2} - \Delta'_{j'_3}$, etc. The relevant coefficients are

$$C'_{j'_1,j'_2,j'_3} = \sqrt{\frac{\gamma(\frac{1}{k'+2})}{\gamma(\frac{2j'_1+1}{k'+2})\gamma(\frac{2j'_2+1}{k'+2})\gamma(\frac{2j'_3+1}{k'+2})}}\,\frac{P(j'_1+j'_2+j'_3+1)\,P(j'_{12})\,P(j'_{23})\,P(j'_{31})}{P(2j'_1)\,P(2j'_2)\,P(2j'_3)} \tag{A.18}$$

and

$$P(j') = \prod_{m=1}^{j'}\gamma(\tfrac{m}{k'+2})\,,\qquad P(0)=1\,,\qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{A.19}$$

The functions $P(j)$ are nonvanishing for $0 \le j' \le k'+1$. Therefore, $C'_{j'_1,j'_2,j'_3} \neq 0$, if

$$j'_1 + j'_2 + j'_3 \le k'\,. \tag{A.20}$$

In m'-space the three-point function can be written as

$$\langle \Phi'_{j'_1,m'_1,\bar m'_1}(z_1,\bar z_1)\Phi'_{j'_2,m'_2,\bar m'_2}(z_2,\bar z_2)\Phi'_{j'_3,m'_3,\bar m'_3}(z_3,\bar z_3)\rangle = \delta^2(\textstyle\sum_{a=1}^3 m'_a)\,\mathcal{D}^{j'_3}_{j'_1j'_2}\displaystyle\prod_{i<j}\frac{1}{|z_{ij}|^{2\Delta'_{ij}}}\,, \tag{A.21}$$

where the SU(2) operator algebra coefficients $\mathcal{D}^{j'_3}_{j'_1j'_2}$ can be found in [17] for the case that

$$m'_1 = \bar m'_1 = j'_1 - d\,,\qquad m'_2 = \bar m'_2 = j'_2\,,\qquad m'_3 = \bar m'_3 = -j'_3 = -(j'_1+j'_2-d)\,. \tag{A.22}$$

They are given by (see eq. (2.46) in [17])

$$\mathcal{D}^{j'_3}_{j'_1j'_2} = \frac{(2j'_2)!\,j'_{13}!}{j'_{12}!\,(2j'_3)!}\prod_{i=1}^{d}\frac{\gamma(\frac{i}{k'+2})\,\gamma(\frac{2j'_1+1+i}{k'+2})}{\gamma(1+\frac{2j'_2-i+1}{k'+2})\,\gamma(\frac{2j'_3-i+1}{k'+2})}\sqrt{(a'_{j'_1})^{-1}\,a'_{j'_2}\,a'_{j'_3}} \tag{A.23}$$

with

$$a'_{j'} = \prod_{i=1}^{2j'}\frac{\gamma(1+\frac{i}{k'+2})}{\gamma(\frac{1+i}{k'+2})} \tag{A.24}$$

and $d = j'_{12} = j'_1 + j'_2 - j'_3$. Here we performed a cyclic rotation of the indices, $j'_1 \to j'_2$, etc. Of course, the coefficients $\mathcal{D}^{j'_3}_{j'_1j'_2}$ are related to the structure constants $C'_{j'_1,j'_2,j'_3}$ given by eq. (A.18). After some algebra, one finds

$$\mathcal{D}^{j'_3}_{j'_1j'_2} = \frac{\Gamma(j'_{13}+1)\,\Gamma(j'_{12}+1)}{\Gamma(2j'_1+1)}\,C'_{j'_1,j'_2,j'_3} \equiv \hat W(j'_i,m'_i)\,C'_{j'_1,j'_2,j'_3}\,. \tag{A.25}$$

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B. The correlators G_3^{AA}, G_3^{AB}, G_3^{BA}

In this section we give some further details on the computation of the terms G_3^{AA}, G_3^{AB} and G_3^{BA} appearing in the worldsheet three-point function G_3 in section 3.2.2. These terms can be written more explicitly by substituting the operators $\mathcal{W}_{j',m',m}$ and $\mathcal{W}^0_{j',m',m}$, as defined in eqs. (3.10) and (3.17), into eq. (3.31).
For the term G_3^{AA}, we then get

$$\begin{aligned} G_3^{AA} = & \, g_s^{-2} b_{a_1} b_{a_2} b_{a_3} b_{\bar a_1} b_{\bar a_2} b_{\bar a_3} \langle \Phi'_{j'_1,m'_1,\bar m'_1} \Phi'_{j'_2,m'_2,\bar m'_2} \Phi'_{j'_3,m'_3,\bar m'_3} \rangle \, |z_{12}|^{-2} \\ & \times \langle : \psi^{a_1} \bar\psi^{\bar a_1} \Phi_{j_1,m_1-a_1,\bar m_1 - \bar a_1} :: \psi^{a_2} \bar\psi^{\bar a_2} \Phi_{j_2,m_2-a_2,\bar m_2-\bar a_2} :: J^{a_3} \bar J^{\bar a_3} \Phi_{j_3,m_3-a_3,\bar m_3-\bar a_3} : \rangle \, , \end{aligned} \tag{B.1}$$

where we have to sum over a_i and $\bar a_i$ ($i = 1, 2, 3$). The factor $|z_{12}|^{-2}$ is again the contribution from the ghosts. Using Wick contraction and the OPE's (3.4), G_3^{AA} can be rewritten as

$$\begin{aligned} G_3^{AA} = & \, g_s^{-2} b_{a_1} b_{a_2} b_{a_3} b_{\bar a_1} b_{\bar a_2} b_{\bar a_3} \langle \Phi'_{j'_1,m'_1,\bar m'_1} \Phi'_{j'_2,m'_2,\bar m'_2} \Phi'_{j'_3,m'_3,\bar m'_3} \rangle \frac{k^2}{4} \frac{\eta^{a_1a_2}\eta^{\bar a_1 \bar a_2}}{|z_{13}|^2 |z_{23}|^2 |z_{12}|^2} \\ & \times \Big(f_{a_3}^{m_1-a_1} f_{\bar a_3}^{\bar m_1 - \bar a_1} \langle \Phi_{j_1,m_1-a_1+a_3,\bar m_1-\bar a_1+\bar a_3} \Phi_{j_2,m_2-a_2,\bar m_2-\bar a_2} \Phi_{j_3,m_3-a_3,\bar m_3-\bar a_3} \rangle \\ & \quad + f_{a_3}^{m_2-a_2} f_{\bar a_3}^{\bar m_2-\bar a_2} \langle \Phi_{j_1,m_1-a_1,\bar m_1-\bar a_1} \Phi_{j_2,m_2-a_2+a_3,\bar m_2-\bar a_2+\bar a_3} \Phi_{j_3,m_3-a_3,\bar m_3-\bar a_3} \rangle \\ & \quad + f_{a_3}^{m_1-a_1} f_{\bar a_3}^{\bar m_2-\bar a_2} \langle \Phi_{j_1,m_1-a_1+a_3,\bar m_1-\bar a_1} \Phi_{j_2,m_2-a_2,\bar m_2-\bar a_2+\bar a_3} \Phi_{j_3,m_3-a_3,\bar m_3-\bar a_3} \rangle \\ & \quad + f_{a_3}^{m_2-a_2} f_{\bar a_3}^{\bar m_1-\bar a_1} \langle \Phi_{j_1,m_1-a_1,\bar m_1-\bar a_1+\bar a_3} \Phi_{j_2,m_2-a_2+a_3,\bar m_2-\bar a_2} \Phi_{j_3,m_3-a_3,\bar m_3-\bar a_3} \rangle \Big) \, , \end{aligned} \tag{B.2}$$

with $f_a^m = (m-j+1, m, m+j-1)$ for $(a = +, 0, -)$. Due to the constraints $a_1 = -a_2$ and $\bar a_1 = -\bar a_2$, we effectively sum only over four different indices ($a_1, a_3, \bar a_1, \bar a_3$ say). We therefore get $3^4 = 81$ terms which we sum up in the same way as explained in detail for G_3^{BB} in section 3.2.2. We find

$$G_3^{AA} = g_s^{-2} g^{AA}(j'_i)\, C_{j_1,j_2,j_3} C'_{j'_1,j'_2,j'_3} W(j_i-1;m_i) \hat W(j'_i;m'_i) \prod_{i<j} \frac{1}{|z_{ij}|^{2(\Delta_{ij}+\Delta'_{ij}+1)}} \, , \tag{B.3}$$

with

$$g^{AA}(j'_i) = \frac{k^2}{16} (1 + j'_1 + j'_2 - j'_3)^2 \, . \tag{B.4}$$

Next, we consider the correlator G_3^{AB} given by

$$\begin{aligned} G_3^{AB} = & \, g_s^{-2} \frac{2}{k} b_{a_1} b_{a_2} b_{a_3} b_{\bar a_1} b_{\bar a_2} A_{\bar a_3 \bar b_3} \langle \Phi'_{j'_1,m'_1,\bar m'_1} \Phi'_{j'_2,m'_2,\bar m'_2} \Phi'_{j'_3,m'_3,\bar m'_3} \rangle \langle \bar\psi^{\bar a_1} \bar\psi^{\bar a_2} \bar\psi^{\bar a_3} \bar\psi^{\bar b_3} \rangle \, |z_{12}|^{-2} \\ & \times \langle : \psi^{a_1} \Phi_{j_1,m_1-a_1,\bar m_1-\bar a_1} :: \psi^{a_2} \Phi_{j_2,m_2-a_2,\bar m_2-\bar a_2} :: J^{a_3} \Phi_{j_3,m_3-a_3,\bar m_3-\bar a_3-\bar b_3} : \rangle \, . \end{aligned} \tag{B.5}$$

With the help of the OPE's (3.4) and eq. (3.33), we get

$$\begin{aligned} G_3^{AB} = & \, g_s^{-2} b_{a_1} b_{a_2} b_{a_3} b_{\bar a_1} b_{\bar a_2} A_{\bar a_3 \bar b_3} \langle \Phi'_{j'_1,m'_1,\bar m'_1} \Phi'_{j'_2,m'_2,\bar m'_2} \Phi'_{j'_3,m'_3,\bar m'_3} \rangle k^2 \frac{\eta^{\bar a_1 [\bar b_3} \eta^{\bar a_3] \bar a_2}}{|z_{13}|^2 |z_{23}|^2 |z_{12}|^2} \eta^{a_1 a_2} \\ & \times \Big(f_{a_3}^{m_1-a_1} \langle \Phi_{j_1,m_1-a_1+a_3,\bar m_1-\bar a_1} \Phi_{j_2,m_2-a_2,\bar m_2-\bar a_2} \Phi_{j_3,m_3-a_3,\bar m_3-\bar a_3-\bar b_3} \rangle \\ & \quad + f_{a_3}^{m_2-a_2} \langle \Phi_{j_1,m_1-a_1,\bar m_1-\bar a_1} \Phi_{j_2,m_2-a_2+a_3,\bar m_2-\bar a_2} \Phi_{j_3,m_3-a_3,\bar m_3-\bar a_3-\bar b_3} \rangle \Big) \, , \end{aligned} \tag{B.6}$$

with $f_a^m = (m-j, m, m+j)$ for $(a = +, 0, -)$, as before. Here we have to sum over the seven indices $a_i, \bar{a}_i$, and b_3 $(i = 1,2,3)$. Again, due to several constraints,

$$
\begin{gathered}
a_1 = -a_2 \,, \\
(\bar{a}_1 = -\bar{b}_3 \,, \quad \bar{a}_2 = -\bar{a}_3 \,, \quad \bar{a}_3 \neq \bar{b}_3) \qquad \text{or} \qquad (\bar{a}_1 = -\bar{a}_3 \,, \quad \bar{a}_2 = -\bar{b}_3 \,, \quad \bar{a}_3 \neq \bar{b}_3) \,,
\end{gathered}
\tag{B.7}
$$

there are only 108 nonvanishing terms (out of 3^7). The summation of these terms yields

$$
G_3^{AB} = g_s^{-2} g^{AB}(j_i')\, C_{j_1,j_2,j_3} C'_{j_1',j_2',j_3'} W(j_i-1; m_i) \hat{W}(j_i'; m_i') \prod_{i<j} \frac{1}{|z_{ij}|^{2(\Delta_{ij}+\Delta'_{ij}+1)}} \,, \tag{B.8}
$$

with

$$
g^{AB}(j_i') = \frac{k^2}{16}\,(1 + j_1' + j_2' - j_3') 2 j_3' \,. \tag{B.9}
$$

The same result is obtained for G_3^{BA} (i.e. $G_3^{BA} = G_3^{AB}$).

C. Product of $\mathrm{SL}(2)_k$ and $\mathrm{SU}(2)_{k'}$ structure constants

In this appendix we compute the product of the $\mathrm{SL}(2)_k$ and $\mathrm{SU}(2)_{k'}$ structure constants. Here we work with the supersymmetric levels, i.e. we need to shift $k \to k+2$ and $k' \to k'-2$ in C_{j_1,j_2,j_3} and $C'_{j_1',j_2',j_3'}$, respectively, and identify $k = k'$. From eq. (A.10) we then find a simple relation between the functions $G(j)$ and $P(j')$ appearing in the SL(2) and SU(2) structure constants,

$$
G(-j) = \frac{1}{\gamma(\frac{j-1}{k})} \cdots \frac{1}{\gamma(\frac{1}{k})} G(-1) = \frac{G(-1)}{P(j-1)} \,. \tag{C.1}
$$

Substituting this into the definition of the SL(2) structure constant C_{j_1,j_2,j_3} yields

$$
\begin{aligned}
C_{j_1,j_2,j_3} &= \frac{1}{(2\pi)^2} \frac{1}{2\pi^3} \frac{k\, P(2j_1-2) P(2j_2-2) P(2j_3-2)}{\nu^{j_1+j_2+j_3-2} P(j_1+j_2+j_3-2) P(j_{12}-1) P(j_{23}-1) P(j_{31}-1)} \\
&= \frac{1}{(2\pi)^3} \sqrt{\frac{\gamma(\frac{1}{k})}{\gamma(\frac{2j_1-1}{k})\gamma(\frac{2j_2-1}{k})\gamma(\frac{2j_3-1}{k})}} \frac{1}{C'_{j_1',j_2',j_3'}} \frac{k}{\pi^2 \nu^{j_1+j_2+j_3-2}} \,,
\end{aligned}
\tag{C.2}
$$

where $j_i' = j_i - 1$. Using the definitions of $B(j)$ and ν, eq. (A.5), we finally get

$$
C_{j_1,j_2,j_3} C'_{j_1',j_2',j_3'} = \sqrt{B(j_1) B(j_2) B(j_3)} \,. \tag{C.3}
$$

The (trivial) numerical coefficient in (C.3) is a consequence of our choice of normalisation for the SL(2) amplitudes — see footnote 9.

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